SO(2n+1) in an SO(2n-3)⊗SU(2)⊗SU(2) basis. II. Detailed study of the symmetric representations of the SO(7) group

1982 J. Phys. A: Math. Gen. 15 2677

(http://iopscience.iop.org/0305-4470/15/9/018)

J. Phys. A: Math. Gen. **15** (1982) 2677–2686. Printed in Great Britain

SO(2*n* + 1) in an SO(2*n* − 3)⊗SU(2)⊗SU(2) basis: II. Detailed study of the symmetric representations of the SO(7) group

G Vanden Berghe, H De Meyer† and P De Wilde‡

Seminarie voor Wiskundige Natuurkunde, RUG Krijgslaan 281–S9, B-9000 Gent, Belgium

Received 22 January 1982

Abstract. The SO(7)↓[SU(2)]^3 reduction is analysed in detail. Explicit forms for the different SU(2) generators and the remaining ones are derived. Using the results of the previous paper, basis states describing the symmetric irreducible unitary representations are introduced. Expressions for the reduced matrix elements of the occurring generators with respect to these basis states are determined.

1. Introduction

The wavefunctions of an *N*-octupole-phonon state are fully classified by seven labels. It is well known that these states may be viewed as symmetric representation states of the unitary group U(7). Four of the necessary labels are usually withheld as the ones related to the Casimir operators of the groups appearing in the chain U(7) ⊃ SO(7) ⊃ SO(3) ⊃ SO(2), i.e. the boson number *N*, the seniority *v*, the angular momentum *l* and its projection *m*. A complete classification of the octupole vibrations of the nucleus has been discussed within this chain by Rohozinski (1978). The supplementary introduced labels are the number of quartets and sextets of phonons coupled to spin zero and a non-physical label defining a residual factor. The states derived in this way are however not orthogonal in the extra indices and have not been normalised. Furthermore, the internal labels are not related to the eigenvalue of an operator.

Some time ago the authors developed a method by which it was possible to construct operators which commute with the Casimir operators of the four above-mentioned groups and which are independent of them. In a set of papers (De Meyer and Vanden Berghe 1980, Vanden Berghe and De Meyer 1980, Vanden Berghe *et al* 1981, De Meyer *et al* 1982b) it has been shown with the use of the shift operator techniques how the eigenvalues of such operators can be deduced in a general way. Up to now one supplementary label could be assigned as the eigenvalue of a scalar shift operator. Further work is in progress in order to construct a second and a third operator whose eigenvalues can be used for the purposes we pursue.

Another angle from which the matter can be looked at consists in studying other subgroup structures of SO(7) than the principal subgroup SO(3). One of the

† Research associate NFWO (Belgium).
‡ Research assistant IWONL (Belgium).

0305-4470/82/092677 + 10$02.00

possibilities is the reduction of SO(7) to $[\mathrm{SU}(2)]^3$. In the previous paper the more general problem, the $\mathrm{SO}(2n+1)\downarrow\mathrm{SO}(2n-3)\otimes\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ reduction, has been discussed. We shall study here the particular case $n=3$. In § 2 a systematic study is made of the relationship between the generators of SO(7), either expressed in the 'natural' basis connected to the $[\mathrm{SU}(2)]^3$ reduction or expressed in the 'physical' basis related to the principal SO(3) subgroup. Special attention is then confined in § 3 to the SO(7) symmetric irreducible unitary representations which are appropriate to the discussion of octupole vibrations of the nuclear surface. In § 4 explicit formulae are derived for the reduced matrix elements of the SO(7) generators which are not belonging to one of the three SU(2) groups, while in § 5 the obtained results are used to calculate the expectation value of the SO(7) second-order Casimir operator.

2. Generators

For physical application it is essential that the irreducible representations of SO(7) can be decomposed into irreducible representations of the principal or physical subgroup SO(3). The SO(7) generators are then grouped as the SO(3) generators l_0, $l_{\pm1}$ together with a seven-dimensional q_μ and an eleven-dimensional p_μ irreducible tensor under SO(3). Considered in spherical tensor form, these generators satisfy the commutation relationships

$$[l_\mu, l_\nu] = -\sqrt{2}\langle 1\mu 1\nu|1\mu+\nu\rangle l_{\mu+\nu}, \tag{2.1}$$

$$[l_\mu, q_\nu] = -2\sqrt{3}\langle 1\mu 3\nu|3\mu+\nu\rangle q_{\mu+\nu}, \tag{2.2}$$

$$[q_\mu, q_\nu] = -(1/2\sqrt{7})\langle 3\mu 3\nu|1\mu+\nu\rangle l_{\mu+\nu} - \langle 3\mu 3\nu|3\mu+\nu\rangle q_{\mu+\nu} + \langle 3\mu 3\nu|5\mu+\nu\rangle p_{\mu+\nu}, \tag{2.3}$$

$$[l_\mu, p_\nu] = -\sqrt{2}\sqrt{3}\sqrt{5}\langle 1\mu 5\nu|5\mu+\nu\rangle p_{\mu+\nu}, \tag{2.4}$$

$$[p_\mu, p_\nu] = -\sqrt{5}\sqrt{11}/(2\sqrt{2}\times 7)\langle 5\mu 5\nu|1\mu+\nu\rangle l_{\mu+\nu} - \sqrt{13}/\sqrt{7}\langle 5\mu 5\nu|5\mu+\nu\rangle p_{\mu+\nu} \tag{2.5}$$

$$[q_\mu, p_\nu] = \sqrt{11}/\sqrt{7}\langle 3\mu 5\nu|3\mu+\nu\rangle q_{\mu+\nu}. \tag{2.6}$$

More explicit formulae for these commutators are given in the appendices of Vanden Berghe and De Meyer (1980) and De Meyer and Vanden Berghe (1980).

In order to make the connection between this set of generators and the ones describing the $\mathrm{SU}(2)\otimes\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ subgroup structure, it is easiest to proceed via the Cartan–Weyl (Wybourne 1974, Cartan 1894) formalism. Briefly, one needs to identify the commuting set H_i $(i=1,2,3)$ and the stepping operators E_α such that $[H_i, E_\alpha]=\alpha_i E_\alpha$. One can choose $H_1=l_0$, $H_2=q_0$ and $H_3=p_0$. In order to obtain the standard form for E_α, it is necessary to define besides $p_{\pm5}$, $p_{\pm4}$ fourteen new operators as linear combinations of $l_{\pm1}$, $q_{\pm1}$, $p_{\pm1}$; $q_{\pm2}$, $p_{\pm2}$ and $q_{\pm3}$, $p_{\pm3}$, i.e.

$$S^{(i)}_{\pm1} = A^{(i)}l_{\pm1} + B^{(i)}q_{\pm1} + C^{(i)}p_{\pm1} \qquad (i=1,2,3),$$

$$P^{(j)}_{\pm2} = D^{(j)}q_{\pm2} + E^{(j)}p_{\pm2} \qquad (j=1,2),$$

$$Q^{(k)}_{\pm3} = F^{(k)}q_{\pm3} + G^{(k)}p_{\pm3} \qquad (k=1,2),$$

such that

$$[q_0, S^{(i)}_{\pm1}] = \pm\sigma^{(i)}S^{(i)}_{\pm1}, \qquad [q_0, P^{(j)}_{\pm2}] = \pm\tau^{(j)}P^{(j)}_{\pm2}, \qquad [q_0, Q^{(k)}_{\pm3}] = \pm\rho^{(k)}Q^{(k)}_{\pm3}.$$

One easily finds the roots: $\sigma^{(1)}=1/\sqrt{6}$, $\sigma^{(2)}=-2/\sqrt{6}$ and $\sigma^{(3)}=0$ with

$$S^{(1)}_{\pm 1}=B^{(1)}[3/14l_{\pm 1}+q_{\pm 1}-\sqrt{5}/\sqrt{7}p_{\pm 1}], \tag{2.7}$$

$$S^{(2)}_{\pm 1}=B^{(2)}[-3/28l_{\pm 1}+q_{\pm 1}+\sqrt{5}/(2\sqrt{7})p_{\pm 1}], \tag{2.8}$$

$$S^{(3)}_{\pm 1}=B^{(3)}[\sqrt{5}/(6\sqrt{7})l_{\pm 1}+p_{\pm 1}], \tag{2.9}$$

$\tau^{(1)}=1/\sqrt{6}$ and $\tau^{(2)}=-2/\sqrt{6}$, with

$$P^{(1)}_{\pm 2}=C^{(1)}[-1/\sqrt{2}q_{\pm 2}+p_{\pm 2}], \tag{2.10}$$

$$P^{(2)}_{\pm 2}=C^{(2)}[\sqrt{2}q_{\pm 2}+p_{\pm 2}], \tag{2.11}$$

$\rho^{(1)}=-1/\sqrt{6}$ and $\rho^{(2)}=2/\sqrt{6}$, with

$$Q^{(1)}_{\pm 3}=D^{(1)}[1/\sqrt{2}q_{\pm 3}+p_{\pm 3}], \tag{2.12}$$

$$Q^{(2)}_{\pm 3}=D^{(2)}[-\sqrt{2}q_{\pm 3}+p_{\pm 3}]. \tag{2.13}$$

The $B^{(i)}$ $(i=1,2,3)$, $C^{(i)}$ $(i=1,2)$ and $D^{(i)}$ $(i=1,2)$ are 'normalisation' factors, which will be determined by supplementary conditions. It is easy to verify that the operators (2.7)–(2.13) also are stepping operators with respect to l_0 and p_0. Starting from this root structure, it is clear that a simple rotation will make the three commuting SU(2) subalgebras manifest. This corresponds to a change of basis from l_0, q_0 and p_0 to a linear combination of these three generators, called s_0, t_0 and u_0, such that

$$\begin{array}{lll}[s_0, S^{(3)}_{\pm 1}]=\pm S^{(3)}_{\pm 1}, & [s_0, Q^{(1)}_{\pm 3}]=0, & [s_0, Q^{(2)}_{\pm 3}]=0\\ [t_0, S^{(3)}_{\pm 1}]=0, & [t_0, Q^{(2)}_{\pm 3}]=\pm Q^{(2)}_{\pm 3}, & [t_0, Q^{(1)}_{\pm 3}]=0\\ [u_0, S^{(3)}_{\pm 1}]=0, & [u_0, Q^{(2)}_{\pm 3}]=0, & [u_0, Q^{(1)}_{\pm 3}]=\pm Q^{(1)}_{3}.\end{array}$$

These conditions yield

$$s_0=\tfrac{1}{28}(l_0+6\sqrt{21}p_0), \tag{2.14}$$

$$t_0=\tfrac{1}{84}(9l_0+28\sqrt{6}q_0-2\sqrt{21}p_0), \tag{2.15}$$

$$u_0=\tfrac{1}{42}(9l_0-14\sqrt{6}q_0-2\sqrt{21}p_0). \tag{2.16}$$

Now we must impose a condition on $s_{\pm 1}=S^{(3)}_{\pm 1}$, $t_{\pm 1}=Q^{(2)}_{\pm 3}$ and $u_{\pm 1}=Q^{(1)}_{\pm 3}$ such that $[s_{+1}, s_{-1}]=-s_0$, $[t_{+1}, t_{-1}]=-t_0$ and $[u_{+1}, u_{-1}]=-u_0$. This yields $B^{(3)}=-3/(\sqrt{2}\sqrt{7})$, $D^{(1)}=-2/\sqrt{3}$, $D^{(2)}=-1/\sqrt{3}$. The three commuting SU(2) subgroups are then generated by the sets $\{s_{+1}, s_0, s_{-1}\}$, $\{t_{+1}, t_0, t_{-1}\}$ and $\{u_{+1}, u_0, u_{-1}\}$. The remaining generators $S^{(1)}_{\pm 1}$, $S^{(2)}_{\pm 1}$, $P^{(1)}_{\pm 2}$ and $P^{(2)}_{\pm 2}$, $p_{\pm 4}$ and $p_{\pm 5}$ form a bispinor-vector $T^{[1/2\ 1/2\ 1]}_{\alpha\ \beta\ \gamma}$ under the $[\mathrm{SU}(2)]^3$ subgroup. The subscripts on the bispinor-vector are s, t, u ordered. That is:

$$[s_\mu, T^{[1/2\ 1/2\ 1]}_{\alpha\ \beta\ \gamma}]=\tfrac{1}{2}\sqrt{3}\langle 1/2\alpha\, 1\mu|1/2\alpha+\mu\rangle T^{[1/2\ 1/2\ 1]}_{\alpha+\mu\ \beta\ \gamma}, \tag{2.17}$$

$$[t_\mu, T^{[1/2\ 1/2\ 1]}_{\alpha\ \beta\ \gamma}]=\tfrac{1}{2}\sqrt{3}\langle 1/2\beta\, 1\mu|1/2\beta+\mu\rangle T^{[1/2\ 1/2\ 1]}_{\alpha\ \beta+\mu\ \gamma}, \tag{2.18}$$

$$[u_\mu, T^{[1/2\ 1/2\ 1]}_{\alpha\ \beta\ \gamma}]=\sqrt{2}\langle 1\gamma\, 1\mu|1\gamma+\mu\rangle T^{[1/2\ 1/2\ 1]}_{\alpha\ \beta\ \gamma+\mu}. \tag{2.19}$$

The bispinor-vector nature is evident from considering the commutators of s_0, t_0 and u_0 with each of the remaining generators. The values for the occurring normalisation factors are fixed by requiring that the remaining commutators involving $s_{\pm 1}$, $t_{\pm 1}$ and

$u_{\pm 1}$ have the appropriate form. Thus, one finds

$$T^{[\ 1/2\ \ 1/2\ \ 1]}_{\pm 1/2\ \pm 1/2\ \pm 1} = \pm p_{\pm 5}, \tag{2.20}$$

$$T^{[\ 1/2\ \ 1/2\ \ 1]}_{\mp 1/2\ \pm 1/2\ \pm 1} = \mp p_{\pm 4}, \tag{2.21}$$

$$T^{[\ 1/2\ \ 1/2\ \ 1]}_{\pm 1/2\ \mp 1/2\ \pm 1} = \mp(1/\sqrt{3})(\sqrt{2}q_{\pm 2} + p_{\pm 2}), \tag{2.22}$$

$$T^{[\ 1/2\ \ 1/2\ 1]}_{\pm 1/2\ \pm 1/2\ 0} = \mp(\sqrt{2}/\sqrt{3})(-q_{\pm 2}/\sqrt{2} + p_{\pm 2}), \tag{2.23}$$

$$T^{[\ 1/2\ \ 1/2\ \ 1]}_{\mp 1/2\ \mp 1/2\ \pm 1} = \pm(\sqrt{2}/\sqrt{3})(-\tfrac{3}{28}l_{\pm 1} + q_{\pm 1} + (\sqrt{5}/2\sqrt{7})p_{\pm 1}), \tag{2.24}$$

$$T^{[\ 1/2\ \ 1/2\ 1]}_{\mp 1/2\ \pm 1/2\ 0} = \mp(1/\sqrt{3})(\tfrac{3}{14}l_{\pm 1} + q_{\pm 1} - (\sqrt{5}/\sqrt{7})p_{\pm 1}). \tag{2.25}$$

Note that the T-generators are so defined that

$$T^{[1/2\ 1/2\ 1]\dagger}_{\alpha\ \ \beta\ \ \gamma} = (-1)^{\alpha+\beta+\gamma} T^{[1/2\ 1/2\ \ 1]}_{-\alpha\ -\beta\ -\gamma}. \tag{2.26}$$

From equations (2.14)–(2.16) we can deduce the primary equation which relates the $[SU(2)]^3$ basis and the SO(7)–SO(3) basis; namely

$$l_0 = s_0 + 3t_0 + 3u_0. \tag{2.27}$$

It is also interesting to deduce from (2.1)–(2.6), taking into account the definitions (2.20)–(2.24), the commutators of the elements of the bispinor-vector among themselves. This gives:

$$\begin{aligned}
&[T^{[1/2\ 1/2\ \ 1\ ]}_{\alpha_1\ \ \beta_1\ \ \gamma_1}, T^{[1/2\ 1/2\ \ 1\ ]}_{\alpha_2\ \ \beta_2\ \ \gamma_2}] \\
&\quad = (1/\sqrt{2})\delta_{\beta_1,-\beta_2}\delta_{\gamma_1,-\gamma_2}(-1)^{3/2-\beta_1-\gamma_1}\langle 1/2\alpha_1\ 1/2\alpha_2|1\alpha_1+\alpha_2\rangle s_{\alpha_1+\alpha_2} \\
&\qquad + (1/\sqrt{2})\delta_{\alpha_1,-\alpha_2}\delta_{\gamma_1,-\gamma_2}(-1)^{3/2-\alpha_1-\gamma_1}\langle 1/2\beta_1\ 1/2\beta_2|1\beta_1+\beta_2\rangle t_{\beta_1+\beta_2} \\
&\qquad + (1/\sqrt{2})\delta_{\alpha_1-\alpha_2}\delta_{\beta_1,-\beta_2}(-1)^{1-\alpha_1-\beta_1}\langle 1\gamma_1\ 1\gamma_2|1\gamma_1+\gamma_2\rangle u_{\gamma_1+\gamma_2}.
\end{aligned} \tag{2.28}$$

3. Basis states and recursion relations for reduced matrix elements

In the previous paper (De Meyer *et al* 1982a) it has been proven that each symmetric irreducible representation $[v, 0, 0]$ of SO(7) is fully reduced with respect to the product subgroup SU(2)⊗SU(2)⊗SU(2). Therefore the corresponding state vectors will bear the labels $|su\lambda\mu\nu\rangle$, where $s(s+1)$, $u(u+1)$, λ, μ, ν are the eigenvalues of s^2 or t^2, u^2, s_0, t_0 and u_0 respectively, since it has been shown that for this kind of representation

$$s = t = 0, 1/2, 1, \ldots, v/2 \tag{3.1}$$

and

$$u = v - 2s, v - 2s - 2, \ldots, 1 \text{ or } 0. \tag{3.2}$$

This reduction rule (3.1), (3.2) is not valid for general SO(7) irreducible representations. There one needs for a complete classification of the representations, besides the v quantum number, eight additional labels instead of the five introduced here. The symmetric representations, which are closely related to the octupole-phonon state vectors occurring in the nuclear collective model, are uniquely described by rule (3.1)–(3.2), i.e. within this reduction no degenerate states occur. It is evident that by combining the rules (3.1), (3.2) with the relation (2.27) one can determine the possible values of l and l_0 occurring in the SO(7)–SO(3) basis within the representation $[v\ 0\ 0]$

by a method similar to the well known derivation of the Clebsch–Gordan series for SO(3). This specific property will be analysed and used in more detail in a forthcoming paper.

In order to derive explicit expressions for the reduced matrix elements of the $T^{[1/2\ 1/2\ 1]}$ operators for our choice of basis states $|su\lambda\mu\nu\rangle$, let us consider the linear combinations of these commutators with vector coupling coefficients of the type $[T^{[1/2\ 1/2\ 1]}, T^{[1/2\ 1/2\ 1]}]^{[pqr]}$ with $\{p, q, r\}$ respectively equal to $\{1, 0, 0\}$, $\{0, 1, 0\}$, $\{0, 0, 1\}$, $\{1, 0, 2\}$, $\{0, 1, 2\}$ and $\{1, 1, 1\}$. Clearly one has

$$[T^{[1/2\ 1/2\ 1]}, T^{[1/2\ 1/2\ 1]}]^{[1\ 0\ 0]}_{\mu\ 0\ 0} = \lambda s_\mu,$$

and in order to find λ, we need merely consider one component, say $\mu = -1$; this yields $\lambda = \sqrt{3}$. In an analogous way one finds

$$[T^{[1/2\ 1/2\ 1]}, T^{[1/2\ 1/2\ 1]}]^{[0\ 1\ 0]}_{0\ \mu\ 0} = \sqrt{3} t_\mu,$$

$$[T^{[1/2\ 1/2\ 1]}, T^{[1/2\ 1/2\ 1]}]^{[0\ 0\ 1]}_{0\ 0\ \mu} = \sqrt{2} u_\mu,$$

$$\begin{aligned}[T^{[1/2\ 1/2\ 1]}, T^{[1/2\ 1/2\ 1]}]^{[1\ 0\ 2]}_{\mu\ 0\ \nu} &= [T^{[1/2\ 1/2\ 1]}, T^{[1/2\ 1/2\ 1]}]^{[0\ 1\ 2]}_{0\ \mu\ \nu} \\ &= [T^{[1/2\ 1/2\ 1]}, T^{[1/2\ 1/2\ 1]}]^{[1\ 1\ 1]}_{\mu\ \nu\ \tau} = 0.\end{aligned}$$

Using the relation between these left-hand side objects and the tensor products of two T-operators, for the values of p, q, r considered above, one finds

$$[T^{[1/2\ 1/2\ 1]}, T^{[1/2\ 1/2\ 1]}]^{[p\ q\ r]}_{\alpha\ \beta\ \gamma} = 2(T^{[1/2\ 1/2\ 1]} T^{[1/2\ 1/2\ 1]})^{[p\ q\ r]}_{\alpha\ \beta\ \gamma},$$

so that

$$(T^{[1/2\ 1/2\ 1]} T^{[1/2\ 1/2\ 1]})^{[1\ 0\ 0]}_{\mu\ 0\ 0} = \tfrac{1}{2}\sqrt{3} s_\mu, \tag{3.3}$$

$$(T^{[1/2\ 1/2\ 1]} T^{[1/2\ 1/2\ 1]})^{[0\ 1\ 0]}_{0\ \mu\ 0} = \tfrac{1}{2}\sqrt{3} t_\mu, \tag{3.4}$$

$$(T^{[1/2\ 1/2\ 1]} T^{[1/2\ 1/2\ 1]})^{[0\ 0\ 1]}_{0\ 0\ \mu} = \tfrac{1}{2}\sqrt{2} u_\mu, \tag{3.5}$$

$$(T^{[1/2\ 1/2\ 1]} T^{[1/2\ 1/2\ 1]})^{[1\ 0\ 2]}_{\mu\ 0\ \nu} = (T^{[1/2\ 1/2\ 1]} T^{[1/2\ 1/2\ 1]})^{[0\ 1\ 2]}_{0\ \mu\ \nu} = 0, \tag{3.6}$$

$$(T^{[1/2\ 1/2\ 1]} T^{[1/2\ 1/2\ 1]})^{[1\ 1\ 1]}_{\mu\ \nu\ \tau} = 0. \tag{3.7}$$

Let us now restrict ourselves to the symmetric irreducible representations $[v\ 0\ 0]$ of SO(7) and the corresponding $[\mathrm{SU}(2)]^3$ basis states $|su\lambda\mu\nu\rangle$. We consider the matrix elements between the states $|su\lambda\mu\nu\rangle$ and $|s'u'\lambda'\mu'\nu'\rangle$ of the right-hand and left-hand sides of equations (3.3), (3.5), (3.6) and (3.7). Since for our particular choice of basis states we have $s = t$ (see (3.1)), equation (3.4) and one of the equations (3.6) are redundant. Applying to each of these matrix elements the Wigner–Eckart theorem, one finds by using equation (15.23) of De Shalit and Talmi (1963) and the reduced matrix elements of s and $\boldsymbol{u}$

$$\begin{aligned}(-1)^{u+u'+1} \sum_{s''u''} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ s & s' & s'' \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ s & s' & s'' \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 0 \\ u & u' & u'' \end{Bmatrix} \langle s'u' \| s''u'' \rangle \langle s''u'' \| su \rangle \\ = \tfrac{1}{2}(2s+1)[s(s+1)(2u+1)]^{1/2} \delta_{ss'} \delta_{uu'},\end{aligned} \tag{3.8}$$

$$\begin{aligned}\sqrt{3}(-1)^{u+u'+1} \sum_{s''u''} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ s & s' & s'' \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ s & s' & s'' \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ u & u' & u'' \end{Bmatrix} \langle s'u' \| s''u'' \rangle \langle s''u'' \| su \rangle \\ = \tfrac{1}{2}\sqrt{2}(2s+1)[u(u+1)(2u+1)]^{1/2} \delta_{ss'} \delta_{uu'},\end{aligned} \tag{3.9}$$

$$\sum_{s''u''}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ s & s' & s''\end{Bmatrix}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ s & s' & s''\end{Bmatrix}\begin{Bmatrix}1 & 1 & 2\\ u & u' & u''\end{Bmatrix}\langle s'u'\|s''u''\rangle\langle s''u''\|su\rangle=0, \tag{3.10}$$

$$\sum_{s''u''}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ s & s' & s''\end{Bmatrix}\begin{Bmatrix}\frac{1}{2} & \frac{1}{2} & 1\\ s & s' & s''\end{Bmatrix}\begin{Bmatrix}1 & 1 & 1\\ u & u' & u''\end{Bmatrix}\langle s'u'\|s''u''\rangle\langle s''u''\|su\rangle=0, \tag{3.11}$$

where we have abbreviated $\langle s'u'\|T^{[1/2\ 1/2\ 1]}\|su\rangle$ by $\langle s'u'\|su\rangle$.

From the properties of the $6j$ symbols on the left-hand side of (3.8) it follows that $s'=s$ and $u'=u$. The summation indices can then take on the values $s''=s\pm 1/2$ and $u''=u\pm 1$ or u. However, because of the intimate relation which exists due to (3.1)–(3.2) between u and s on the one side and s'' and u'' on the other side, it is clear that if the values s and u properly describe a symmetric $[\mathrm{SU}(2)]^{(3)}$ representation, the values $s''=s\pm 1/2$ and $u''=u$ do not. This means that in the sum on the left-hand side of (3.8) only four terms have to be withheld.

This yields, when writing out the explicit formulae for the occurring $6j$ symbols,

$$\begin{aligned}(s+1)[\langle su\|s-1/2u+1\rangle\langle s-1/2u+1\|su\rangle+\langle su\|s-1/2u-1\rangle\langle s-1/2u-1\|su\rangle]\\ -s[\langle su\|s+1/2u+1\rangle\langle s+1/2u+1\|su\rangle\\ +\langle su\|s+1/2u-1\rangle\langle s+1/2u-1\|su\rangle]\\ =3s(s+1)(2s+1)^2(2u+1).\end{aligned} \tag{3.12}$$

Due to (2.26) it follows that

$$\langle s'u'\|su\rangle^*=(-1)^{2s-2s'+u-u'}\langle su\|s'u'\rangle. \tag{3.13}$$

Taking this property into account and defining $r=2s$, (3.12) becomes

$$\begin{aligned}(r+2)[|\langle ru\|r-1u+1\rangle|^2+|\langle ru\|r-1u-1\rangle|^2]-r[|\langle r+1u+1\|ru\rangle|^2+|\langle r+1u-1\|ru\rangle|^2]\\ =\tfrac{3}{2}r(r+1)^2(r+2)(2u+1).\end{aligned} \tag{3.14}$$

Similarly in (3.9) one finds $s'=s$, $u'=u$ and $s''=s\pm 1/2$, $u''=u\pm 1$ to yield

$$\begin{aligned}-u[|\langle r+1u+1\|ru\rangle|^2+|\langle ru\|r-1u+1\rangle|^2]+(u+1)[|\langle r+1u-1\|ru\rangle|^2+|\langle ru\|r-1u-1\rangle|^2]\\ =2(r+1)^2u(u+1)(2u+1).\end{aligned} \tag{3.15}$$

In a similar way one can deduce that (3.10) gives rise to two different relations, one based on $s'=s$, $u'=u\pm 2$, $s''=s\pm 1/2$, $u''=u\pm 1$ and the other following from the parameter choice $s'=s$, $u'=u$, $s''=s\pm 1/2$, $u''=u\pm 1$, i.e.

$$r\langle ru\pm 2\|r+1u\pm 1\rangle\langle r+1u\pm 1\|ru\rangle-(r+2)\langle ru\pm 2\|r-1u\pm 1\rangle\langle r-1u\pm 1\|ru\rangle=0, \tag{3.16}$$

and

$$\begin{aligned}r[u(2u-1)|\langle r+1u+1\|ru\rangle|^2+(u+1)(2u+3)|\langle r+1u-1\|ru\rangle|^2]\\ -(r+2)[u(2u-1)|\langle ru\|r-1u+1\rangle|^2\\ +(u+1)(2u+3)|\langle ru\|r-1u-1\rangle|^2]=0.\end{aligned} \tag{3.17}$$

From (3.11) again two relations can be derived, i.e. for $s'=s$, $u'=u$, $s''=s\pm 1/2$ and $u''=u\pm 1$ and for $s'=s\pm 1$, $u'=u$, $s''=s\pm 1/2$ and $u''=u\pm 1$ yielding

$$\begin{aligned}r^2[u|\langle r+1u+1\|ru\rangle|^2-(u+1)|\langle r+1u-1\|ru\rangle|^2]\\ +(r+2)^2[u|\langle ru\|r-1u+1\rangle|^2-(u+1)|\langle ru\|r-1u-1\rangle|^2]=0\end{aligned} \tag{3.18}$$

and

$$r[u\langle r+2u\|r+1u+1\rangle\langle r+1u+1\|ru\rangle-(u+1)\langle r+2u\|r+1u-1\rangle\langle r+1u-1\|ru\rangle]$$
$$+(r+2)[u\langle r-2u\|r-1u+1\rangle\langle r-1u+1\|ru\rangle$$
$$-(u+1)\langle r-2u\|r-1u-1\rangle\langle r-1u-1\|ru\rangle]=0. \tag{3.19}$$

In equations (3.12), (3.14)–(3.19) four types of reduced matrix elements occur, i.e.

$$\langle r+1u+1\|ru\rangle=A(r,u), \qquad \langle r+1u-1\|ru\rangle=B(r,u),$$
$$\langle r-1u+1\|ru\rangle=C(r,u), \qquad \langle r-1u-1\|ru\rangle=D(r,u).$$

Due to (3.13) one finds $C(r,u)=B^*(r-1,u+1)$ and $D(r,u)=A^*(r-1,u-1)$, so that all previous relations are induction equations for $A(r,u)$ and $B(r,u)$ which read as follows:

$$(r+2)(|B(r-1,u+1)|^2+|A(r-1,u-1|^2)-r(|A(r,u)|^2+|B(r,u)|^2)$$
$$=\tfrac{3}{2}r(r+1)^2(r+2)(2u+1), \tag{3.20}$$

$$-u(|A(r,u)|^2+|B(r-1,u+1)|^2)+(u+1)(|B(r,u)|^2+|A(r-1,u-1)|^2)$$
$$=2(r+1)^2u(u+1)(2u+1), \tag{3.21}$$

$$rB^*(r,u+2)A(r,u)-(r+2)A(r-1,u+1)B^*(r-1,u+1)=0, \tag{3.22a}$$

$$rA^*(r,u-2)B(r,u)-(r+2)B(r-1,u-1)A^*(r-1,u-1)=0, \tag{3.22b}$$

$$r[u(2u-1)|A(r,u)|^2+(u+1)(2u+3)|B(r,u)|^2]-(r+2)[u(2u-1)|B(r-1,u+1)|^2$$
$$+(u+1)(2u+3)|A(r-1,u-1)|^2]=0, \tag{3.23}$$

$$r^2[u|A(r,u)|^2-(u+1)|B(r,u)|^2]+(r+2)^2[u|B(r-1,u+1)|^2$$
$$-(u+1)|A(r-1,u-1)|^2]=0, \tag{3.24}$$

$$r[uB(r+1,u+1)A(r,u)-(u+1)A(r+1,u-1)B(r,u)]$$
$$+(r+2)[uA^*(r-2,u)B^*(r-1,u+1)$$
$$-(u+1)B^*(r-2,u)A^*(r-1,u-1)]=0. \tag{3.25}$$

4. Solution of the induction equations

From equations (3.20), (3.21) and (3.23), one can solve three of the occurring squares in terms of the fourth; this yields

$$|B(r-1,u+1)|^2=r/(r+2)|A(r,u)|^2+\tfrac{1}{2}r(r+1)^2(u+1)(2u+3), \tag{4.1}$$

$$|A(r-1,u-1)|^2=ur/[(u+1)(r+2)]|A(r,u)|^2+\tfrac{1}{2}ur(r+1)(2r+2u+3), \tag{4.2}$$

$$|B(r,u)|^2=u/(u+1)|A(r,u)|^2+\tfrac{1}{2}u(2u+1)(r+1)(r+2)^2. \tag{4.3}$$

From (4.1) and (4.3) it follows that

$$|B(r,u+2)|^2=(r+1)/(r+3)|A(r+1,u+1)|^2+1/2(r+1)(r+2)^2(u+2)(2u+5)$$
$$=(u+2)/(u+3)|A(r,u+2)|^2+1/2(r+1)(r+2)^2(u+2)(2u+5).$$

This shows that

$$\frac{r}{r+2}|A(r,u)|^2=\frac{u+1}{u+2}|A(r-1,u+1)|^2 \tag{4.4}$$

and (4.2) can be written as

$$\frac{(u+1)(r+1)}{(u+2)(r+3)}|A(r+1,u+1)|^2 = |A(r,u)|^2-\tfrac{1}{2}(u+1)(r+1)(r+2)(2r+2u+7). \tag{4.5}$$

Substituting now

$$|A(r,u)|^2=(u+1)(r+1)(r+2)X(r,u) \tag{4.6}$$

into (4.4) and (4.5) yields

$$X(r-1,u+1)=X(r,u) \tag{4.7}$$

and

$$X(r+1,u+1)=X(r,u)-\tfrac{1}{2}(2r+2u+7). \tag{4.8}$$

Combining these two equations leads to

$$X(r+2,u)=X(r,u)-\tfrac{1}{2}(2r+2u+7) \tag{4.9}$$

and

$$X(r,u+2)=X(r,u)-\tfrac{1}{2}(2r+2u+7). \tag{4.10}$$

The solution $X(r,u)$ satisfying equations (4.9) and (4.10) and the boundary condition $X(r,u)=0$ for $r+u=v$, which is a consequence of (3.2) and the explicit definition of $X(r,u)$, can be written as

$$X(r,u)=\tfrac{1}{4}(v-u-r)(v+u+r+5)$$

or

$$|A(r,u)|^2=\tfrac{1}{4}(u+1)(r+1)(r+2)(v-u-r)(v+u+r+5). \tag{4.11}$$

Substituting this result into (4.3) yields

$$|B(r,u)|^2=\tfrac{1}{4}u(r+1)(r+2)(v+u-r+1)(v-u+r+4). \tag{4.12}$$

The reader can now easily convince himself of the fact that (4.11) and (4.12) are also solutions of equations (3.22), (3.24) and (3.25) which are redundant for finding explicit expressions for A and B. The solution to the induction equations is completed once we adopt the phase convention that the reduced matrix elements are the positive square roots of $|A(r,u)|^2$ and $|B(r,u)|^2$, giving

$$\langle r+1\, u+1\|T^{[1/2\ 1/2\ 1]}\|ru\rangle=\tfrac{1}{2}[(u+1)(r+1)(r+2)(v-u-r)(v+u+r+5)]^{1/2}, \tag{4.13}$$

$$\langle r+1\, u-1\|T^{[1/2\ 1/2\ 1]}\|ru\rangle=\tfrac{1}{2}[u(r+1)(r+2)(v+u-r+1)(v-u+r+4)]^{1/2}. \tag{4.14}$$

5. The use of the reduced matrix elements

The second-order Casimir operator is in general defined as

$$I_2=\sum_{\rho\sigma} g^{\rho\sigma}\chi_\rho\chi_\sigma\,.$$

where χ_ρ and χ_σ are the generators of the group and $g^{\rho\sigma}$ the reciprocal metric tensor, whereby the metric tensor is defined in terms of the structure constants. For the group SO(7) in the reduction to $[SU(2)]^3$, the structure constants are either given in the right-hand sides of (2.17)–(2.19) and (2.28) or are the well-known SU(2) structure constants. By this, I_2 for the SO(7) group can be written as follows:

$$\begin{aligned} I_2 = & -2T^{[1/2\,1/2\,1]}_{1/2\,1/2\,1}T^{[\,1/2\,1/2\,1]}_{-1/2\,-1/2\,-1}+2T^{[1/2\,1/2\,1]}_{1/2\,1/2\,0}T^{[\,1/2\,1/2\,1]}_{-1/2\,-1/2\,0}-2T^{[1/2\,1/2\,1]}_{1/2\,1/2\,-1}T^{[\,1/2\,1/2\,1]}_{-1/2\,-1/2\,1} \\ & +2T^{[1/2\,1/2\,1]}_{1/2\,-1/2\,1}T^{[\,1/2\,1/2\,1]}_{-1/2\,1/2\,-1}-2T^{[1/2\,1/2\,1]}_{1/2\,-1/2\,0}T^{[\,1/2\,1/2\,1]}_{-1/2\,1/2\,0} \\ & +2T^{[1/2\,1/2\,1]}_{1/2\,-1/2\,-1}T^{[\,1/2\,1/2\,1]}_{-1/2\,1/2\,1}+3s_0-s_0(s_0-1)+2s_{+1}s_{-1}+2t_{+1}t_{-1} \\ & -t_0(t_0-1)+u_{+1}u_{-1}-\tfrac{1}{2}u_0(u_0-1), \end{aligned}$$

or this can be written in a short-hand notation as

$$I_2=-2\sqrt{3}(T^{[1/2\,1/2\,1]}T^{[1/2\,1/2\,1]})^{[0\,0\,0]}_{0\,0\,0}-\tfrac{1}{2}u^2-s^2-t^2. \tag{5.1}$$

The expectation value of this operator with respect to the basis states $|su\lambda\mu\nu\rangle$ can be written as

$$\begin{aligned} & \langle su\lambda\mu\nu|I_2|su\lambda\mu\nu\rangle \\ & \quad =-2\sqrt{3}\langle su\lambda\mu\nu|(T^{[1/2\,1/2\,1]}T^{[1/2\,1/2\,1]})^{[0\,0\,0]}_{0\,0\,0}|su\lambda\mu\nu\rangle \\ & \qquad -\tfrac{1}{2}u(u+1)-2s(s+1), \end{aligned} \tag{5.2}$$

taking into account that for the considered symmetric irreducible representations $\langle s^2\rangle=\langle t^2\rangle$. The matrix element on the right-hand side of (5.2) reduces by taking into account the Wigner–Eckart theorem to

$$\begin{aligned} & \langle su\lambda\mu\nu|(T^{[1/2\,1/2\,1]}T^{[1/2\,1/2\,1]})^{[0\,0\,0]}_{0\,0\,0}|su\lambda\mu\nu\rangle \\ & \quad =(-1)^{2s-\lambda-\mu+u-\nu}\begin{pmatrix} s & 0 & s \\ -\lambda & 0 & \lambda \end{pmatrix}\begin{pmatrix} s & 0 & s \\ -\mu & 0 & \mu \end{pmatrix}\begin{pmatrix} u & 0 & u \\ -\nu & 0 & \nu \end{pmatrix} \\ & \qquad \times\langle su\|(T^{[1/2\,1/2\,1]}T^{[1/2\,1/2\,1]})^{[0\,0\,0]}\|su\rangle \\ & \quad =(2s+1)^{-1}(2u+1)^{-1/2}\langle su\|(T^{[1/2\,1/2\,1]}T^{[1/2\,1/2\,1]})^{[0\,0\,0]}\|su\rangle. \end{aligned} \tag{5.3}$$

By applying equation (15.23) of De Shalit and Talmi (1963) in order to determine the occurring matrix element, and by taking into account the rule (3.1), (3.2), one finds

$$\begin{aligned} & \langle su\|(T^{[1/2\,1/2\,1]}T^{[1/2\,1/2\,1]})^{[0\;\;0]}\|su\rangle \\ & \quad =\frac{1}{2(2s+1)[3(2u+1)]^{1/2}}[|\langle r+1u+1\|ru\rangle|^2+|\langle r+1u-1\|ru\rangle|^2 \\ & \qquad +|\langle ru\|r-1u-1\rangle|^2+|\langle ru\|r-1u+1\rangle|^2], \end{aligned}$$

which by introducing (4.13) and (4.14) reduces to

$$\begin{aligned} & \langle su\|(T^{[1/2\,1/2\,1]}T^{[1/2\,1/2\,1]})^{[0\,0\,0]}\|su\rangle \\ & \quad =(1/4\sqrt{3})(2s+1)(2u+1)^{1/2}[v(v+5)-u(u+1)-4s(s+1)]. \end{aligned} \tag{5.4}$$

Combining (5.2), (5.3) and (5.4), one deduces that

$$\langle su\lambda\mu\nu|I_2|su\lambda\mu\nu\rangle=-\tfrac{1}{2}v(v+5),$$

a result which is well known (Weber *et al* 1966).

6. Conclusion

The reduction of SO(7) to the $[SU(2)]^3$ subgroup structure which is a particular case of the $SO(2n+1)$ reduction to $SU(2)\otimes SU(2)\otimes SO(2n-3)$ discussed in the previous paper (De Meyer *et al* 1982a), has been analysed here in detail. The explicit form of the different SU(2) generators and the remaining generators which are forming a bispinor-vector are explicitly given in terms of the generators of the principal SO(3) subgroup of SO(7) and the seven- and eleven-dimensional irreducible tensor representations with respect to this SO(3). Using the branching rule for the $SO(7)\downarrow[SU(2)]^3$ reduction of symmetric irreducible representations, which produces an unambiguous state labelling scheme, explicit expressions for the reduced matrix elements of the generators with respect to this basis were derived. It has been shown that these expressions can be very useful in the derivation of expectation values of, for example, SO(7) invariants.

Another important result of this analysis is the relation (2.27) which relates the $[SU(2)]^3$ basis and the SO(7)–SO(3) basis. It permits us to determine in a rather easy way the angular momentum contents of a particular SO(7) representation if the range of the three SU(2) labels is known. For the symmetric representations $[v\ 0\ 0]$ of SO(7) the range of these labels has been fixed in the previous paper, so that for these particular representations the allowed l-values can be determined. Previously this information could only be derived by considering characters, a method which is usually quite involved. The relation (2.27) opens possibilities to define so-called 'intrinsic states' in the sense in which they have been introduced by Williams and Pursey (1968) in their $SO(5)\downarrow SU(2)\otimes SU(2)$ reduction study, and to derive analytical expressions for the reduced matrix elements of q and p with respect to the physical basis states labelled by the angular momentum quantum number. On this subject we hope to report in the near future.

References

Cartan E 1894 *Sur la Structure des Groupes des Transformations Finis et Continus, thesis*, Nony, Paris
De Meyer H, De Wilde P and Vanden Berghe G 1982a *J. Phys. A: Math. Gen.* **15** 2665–76
De Meyer H and Vanden Berghe G 1980 *J. Math. Phys.* **21** 1973–6
De Meyer H, Vanden Berghe G and Van de Jeugt J 1982b *J. Math. Phys.* in press
De Shalit A and Talmi I 1963 *Nuclear Shell Theory* (New York: Academic) p 134
Rohozinski S G 1978 *J. Phys. G: Nucl. Phys.* **4** 1075–98
Vanden Berghe G and De Meyer H 1980 *J. Math. Phys.* **21** 1967–72
Vanden Berghe G, De Meyer H and Van der Jeugt J 1981 *J. Math. Phys.* **22** 2332–7
Weber H J, Huber M G and Greiner W 1966 *Z. Phys.* **190** 25–59
Williams S A and Pursey D L 1968 *J. Math. Phys.* **9** 1230–40
Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)